

Series Reference Guide



Intuitively, a series is an infinite sum: $\sum_{n=k}^{\infty} a_n = \overline{a_k + a_{k+1} + \ldots}$ However, it is not possible to add an infinite string of numbers, so the notion of an infinite sum is meaningless. What then is a series exactly? The precise definition that we use is

$$\sum_{n=k}^{\infty} a_n = \lim_{n \to \infty} (a_k + a_{k+1} + \ldots + a_n)$$

The finite sum $a_k + \ldots + a_n$ is called the n^{th} partial sum of the series and denoted S_n . With this notation, we can also write

$$\sum_{n=k}^{\infty} a_n = \lim_{n \to \infty} S_n.$$

If this limit exists (and is finite), we say the series *converges* to the value of the limit. Otherwise, we say the series *diverges*. Note: We have used k as the starting index here. Usually, k is 0 or 1, but it could be any integer more generally.

Test	Series	Statement of Test	When to Use	Additional Comments
Divergence Test	$\sum a_n$	If $\lim_{n \to \infty} a_n \neq 0$, the series diverges.	When you think or know that $\lim_{n \to \infty} a_n \neq 0$	If $\lim_{n \to \infty} a_n = 0$, the test is inconclusive, and the series may or may not converge.
Telescoping Series	$\sum a_{n+k} - a_n$	For telescoping series, the partial sums are simple enough to compute. So just find S_n and take the limit as $n \to \infty$ to determine if the series converges or diverges.	If you see terms cancelling, it's a telescoping series.	Some algebraic manipulations (e.g. partial fractions, properties of log, etc) may be needed to get a series to be in the form of a telescoping series.
Geometric Series	$\sum_{n=0}^{\infty} ar^n$ or $\sum_{n=1}^{\infty} ar^{n-1}$	If $ r < 1$, the series converges absolutely. Else, the series diverges.	If it only involves constants raised to powers involving n , it's probably a geometric series.	A convergent geometric series converges to: $\frac{1^{\text{st}} \text{ term}}{1-r} = \frac{a}{1-r}$.
<i>p</i> -Series	$\sum \frac{1}{n^p}$	If $p > 1$, the series converges. Else, the series diverges.	When working with a p -series	
Integral Test	$\sum_{\substack{f \text{ is } \\ \text{ eventually: } \\ \text{ positive, } \\ \text{ continuous, } \\ \text{ decreasing } }$	Compute $\int_{n}^{\infty} f(x) dx$. If the integral converges, the series converges. If the integral diverges, the series diverges.	When f satisfies the necessary conditions, and the integral is doable.	If it's not obvious that f is decreasing, compute f' and show that $f' < 0$. If the series converges to S , then $R_n = S - S_n \le \int_n^\infty f(x) dx$.
Direct Comparison	Compare $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n,$ $a_n, b_n \ge 0$	If $a_n \leq b_n$ and $\sum b_n$ converges, then $\sum a_n$ converges. If $b_n \leq a_n$ and $\sum b_n$ diverges, then $\sum a_n$ diverges.	Often useful when $a_n \ge 0$ is a complicated expression involving trig terms.	To get something smaller to compare with, make the numerator smaller or the denominator larger. To get something larger to compare with, make the numerator larger or the denominator smaller. Try to keep the overall size of numerator and denominator the same as in the original.
Limit Comparison	Compare $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n,$ $a_n, b_n \ge 0$	If $\lim_{n\to\infty} \frac{a_n}{b_n}$ exists and is positive and finite, then both series behave the same, i.e. they both converge or they both diverge.	Often useful when $a_n \ge 0$ is a complicated expression involving algebraic or exponential terms.	Given a_n , to find an appropriate b_n to compare with, take a_n and drop the smaller expressions, keeping only the largest expressions.
Alternating Series	$\sum_{n \ge 0}^{n} (-1)^n b_n$	If $\lim_{n \to \infty} b_n = 0$ and b_n is eventually decreasing, the series converges.	When working with an alternating series	If it's not obvious, use derivatives to show b_n is decreasing. If the series converges to S , then $R_n = S - S_n \le b_{n+1}$.
Ratio Test	$\sum a_n$	Compute $\lim_{n \to \infty} \left \frac{a_{n+1}}{a_n} \right = L.$ If $L < 1$, the series converges absolutely. If $L > 1$, the series diverges.	When a_n has: factorials (e.g. $n!, (2n + 1)!, \text{ etc}),$ exponentials (e.g. $2^n, 3^{n+2}, \text{ etc}),$ products (e.g. $2 \cdot 5 \cdot \ldots \cdot 3n + 2, \text{ etc})$	If $L = 1$, the test is inconclusive.
Root Test	$\sum a_n$	Compute $\lim_{n \to \infty} \sqrt[n]{ a_n } = L$. If $L < 1$, the series converges absolutely. If $L > 1$, the series diverges.	When a_n is a bunch of stuff all raised to a power involving n , i.e. $a_n = (b_n)^n$.	If $L = 1$, the test is inconclusive.

Source: Stewart, J. (2020). Calculus, 9e. Cengage Learning.